# Problem 1.25

(a) Check product rule (iv) (by calculating each term separately) for the functions

$$\mathbf{A} = x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}, \quad \mathbf{B} = 3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}.$$

(b) Do the same for product rule (ii).

(c) Do the same for rule (vi).

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**Solution**

## Check Product Rule (ii)

Product rule (ii) is

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}. \tag{ii}$$

Calculate the left side.

$$\begin{aligned}
\nabla(\mathbf{A}\cdot\mathbf{B}) &= \nabla[(x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}})\cdot(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}})] \\
&= \nabla[x(3y) + 2y(-2x) + 3z(0)] \\
&= \nabla(-xy) \\
&= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(-xy) \\
&= \hat{\mathbf{x}}\frac{\partial}{\partial x}(-xy) + \hat{\mathbf{y}}\frac{\partial}{\partial y}(-xy) + \hat{\mathbf{z}}\frac{\partial}{\partial z}(-xy) \\
&= \hat{\mathbf{x}}(-y) + \hat{\mathbf{y}}(-x) + \hat{\mathbf{z}}(0)
\end{aligned}$$

Calculate the first term on the right side.

$$\begin{aligned}
\mathbf{A}\times(\nabla\times\mathbf{B}) &= (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\
&= (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \times \left\{\hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2x)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(3y)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(3y)\right]\right\} \\
&= (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \times \{\hat{\mathbf{x}}[(0) - (0)] - \hat{\mathbf{y}}[(0) - (0)] + \hat{\mathbf{z}}[(-2) - (3)]\} \\
&= (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \times (0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} - 5\hat{\mathbf{z}}) \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} \\
&= -10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}
\end{aligned}$$

Calculate the second term on the right side.

$$\begin{aligned}
\mathbf{B}\times(\nabla\times\mathbf{A}) &= (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} \\
&= (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \times \left\{\hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}(3z) - \frac{\partial}{\partial z}(x)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(x)\right]\right\} \\
&= (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \times \{\hat{\mathbf{x}}[(0)-(0)] - \hat{\mathbf{y}}[(0)-(0)] + \hat{\mathbf{z}}[(0)-(0)]\} \\
&= (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \times (0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 3y & -2x & 0 \\ 0 & 0 & 0 \end{vmatrix} \\
&= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}
\end{aligned}$$

Calculate the third term on the right side, using the formula from part (a) of Prob. 1.22.

$$\begin{aligned}
(\mathbf{A}\cdot\nabla)\mathbf{B} &= \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right) \\
&\quad + \hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right) \\
&\quad + \hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right) \\
&= \hat{\mathbf{x}}\left[x\frac{\partial}{\partial x}(3y) + 2y\frac{\partial}{\partial y}(3y) + 3z\frac{\partial}{\partial z}(3y)\right] \\
&\quad + \hat{\mathbf{y}}\left[x\frac{\partial}{\partial x}(-2x) + 2y\frac{\partial}{\partial y}(-2x) + 3z\frac{\partial}{\partial z}(-2x)\right] \\
&\quad + \hat{\mathbf{z}}\left[x\frac{\partial}{\partial x}(0) + 2y\frac{\partial}{\partial y}(0) + 3z\frac{\partial}{\partial z}(0)\right] \\
&= \hat{\mathbf{x}}[x(0) + 2y(3) + 3z(0)] \\
&\quad + \hat{\mathbf{y}}[x(-2) + 2y(0) + 3z(0)] \\
&\quad + \hat{\mathbf{z}}[x(0) + 2y(0) + 3z(0)] \\
&= \hat{\mathbf{x}}(6y) + \hat{\mathbf{y}}(-2x) + 0\hat{\mathbf{z}}
\end{aligned}$$

Calculate the fourth term on the right side, using the formula from part (a) of Prob. 1.22.

$$\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{A} &= \hat{\mathbf{x}}\left(B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}\right) \\
&\qquad + \hat{\mathbf{y}}\left(B_x\frac{\partial A_y}{\partial x}+B_y\frac{\partial A_y}{\partial y}+B_z\frac{\partial A_y}{\partial z}\right) \\
&\qquad\qquad + \hat{\mathbf{z}}\left(B_x\frac{\partial A_z}{\partial x}+B_y\frac{\partial A_z}{\partial y}+B_z\frac{\partial A_z}{\partial z}\right) \\
&= \hat{\mathbf{x}}\left[3y\frac{\partial}{\partial x}(x)-2x\frac{\partial}{\partial y}(x)+0\frac{\partial}{\partial z}(x)\right] \\
&\qquad + \hat{\mathbf{y}}\left[3y\frac{\partial}{\partial x}(2y)-2x\frac{\partial}{\partial y}(2y)+0\frac{\partial}{\partial z}(2y)\right] \\
&\qquad\qquad + \hat{\mathbf{z}}\left[3y\frac{\partial}{\partial x}(3z)-2x\frac{\partial}{\partial y}(3z)+0\frac{\partial}{\partial z}(3z)\right] \\
&= \hat{\mathbf{x}}\,[3y(1)-2x(0)+0(0)] \\
&\qquad + \hat{\mathbf{y}}\,[3y(0)-2x(2)+0(0)] \\
&\qquad\qquad + \hat{\mathbf{z}}\,[3y(0)-2x(0)+0(3)] \\
&= \hat{\mathbf{x}}(3y)+\hat{\mathbf{y}}(-4x)+0\hat{\mathbf{z}}
\end{aligned}$$

Now that the four terms on the right side are known, add them together.

$$\begin{aligned}
\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A} &= [-10y\hat{\mathbf{x}}+5x\hat{\mathbf{y}}+0\hat{\mathbf{z}}]+[0\hat{\mathbf{x}}+0\hat{\mathbf{y}}+0\hat{\mathbf{z}}] \\
&\qquad + [\hat{\mathbf{x}}(6y)+\hat{\mathbf{y}}(-2x)+0\hat{\mathbf{z}}]+[\hat{\mathbf{x}}(3y)+\hat{\mathbf{y}}(-4x)+0\hat{\mathbf{z}}] \\
&= (-10y+0+6y+3y)\hat{\mathbf{x}}+(5x+0-2x-4x)\hat{\mathbf{y}}+0\hat{\mathbf{z}} \\
&= (-y)\hat{\mathbf{x}}+(-x)\hat{\mathbf{y}}+0\hat{\mathbf{z}}
\end{aligned}$$

Since the left side is equal to the right side, product rule (ii) is verified.

## Check Product Rule (iv)

Product rule (iv) is

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \tag{iv}$$

Calculate the left side.

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\
&= \nabla \cdot [\hat{\mathbf{x}}(0 + 6xz) - \hat{\mathbf{y}}(0 - 9yz) + \hat{\mathbf{z}}(-2x^2 - 6y^2)] \\
&= \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9yz) + \frac{\partial}{\partial z}(-2x^2 - 6y^2) \\
&= (6z) + (9z) + (0) \\
&= 15z
\end{aligned}$$

Calculate the right side.

$$\begin{aligned}
\mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) &= (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} - (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\
&= (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) \cdot (0\hat{\mathbf{x}} - 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) - (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \cdot (0\hat{\mathbf{x}} - 0\hat{\mathbf{y}} - 5\hat{\mathbf{z}}) \\
&= [3y(0) + 2x(0) + 0(0)] - [x(0) + 2y(0) + 3z(-5)] \\
&= 15z
\end{aligned}$$

Since the left side is equal to the right side, product rule (iv) is verified.

## Check Product Rule (vi)

Product rule (vi) is

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}). \tag{vi}$$

Calculate the left side.

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \nabla \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\
&= \nabla \times [\hat{\mathbf{x}}(0 + 6xz) - \hat{\mathbf{y}}(0 - 9yz) + \hat{\mathbf{z}}(-2x^2 - 6y^2)] \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & 9yz & -2x^2 - 6y^2 \end{vmatrix} \\
&= \hat{\mathbf{x}} \left[ \frac{\partial}{\partial y}(-2x^2 - 6y^2) - \frac{\partial}{\partial z}(9yz) \right] - \hat{\mathbf{y}} \left[ \frac{\partial}{\partial x}(-2x^2 - 6y^2) - \frac{\partial}{\partial z}(6xz) \right] + \hat{\mathbf{z}} \left[ \frac{\partial}{\partial x}(9yz) - \frac{\partial}{\partial y}(6xz) \right] \\
&= \hat{\mathbf{x}} [(-12y) - (9y)] - \hat{\mathbf{y}} [(-4x) - (6x)] + \hat{\mathbf{z}} [(0) - (0)] \\
&= \hat{\mathbf{x}}(-21y) + \hat{\mathbf{y}}(10x) + 0\hat{\mathbf{z}}
\end{aligned}$$

Calculate the right side, using the results for $(\mathbf{B} \cdot \nabla)\mathbf{A}$ and $(\mathbf{A} \cdot \nabla)\mathbf{B}$ from before.

$$\begin{aligned}
(\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) &= [\hat{\mathbf{x}}(3y) + \hat{\mathbf{y}}(-4x) + 0\hat{\mathbf{z}}] \\
&\quad - [\hat{\mathbf{x}}(6y) + \hat{\mathbf{y}}(-2x) + 0\hat{\mathbf{z}}] \\
&\quad + (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \left[ \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial y}(-2x) + \frac{\partial}{\partial z}(0) \right] \\
&\quad - (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \left[ \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) \right] \\
&= [\hat{\mathbf{x}}(3y) + \hat{\mathbf{y}}(-4x) + 0\hat{\mathbf{z}}] \\
&\quad - [\hat{\mathbf{x}}(6y) + \hat{\mathbf{y}}(-2x) + 0\hat{\mathbf{z}}] \\
&\quad + (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) [(0) + (0) + (0)] \\
&\quad - (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) [(1) + (2) + (3)] \\
&= [\hat{\mathbf{x}}(3y) + \hat{\mathbf{y}}(-4x) + 0\hat{\mathbf{z}}] - [\hat{\mathbf{x}}(6y) + \hat{\mathbf{y}}(-2x) + 0\hat{\mathbf{z}}] \\
&\quad + (0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) - (18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \\
&= \hat{\mathbf{x}}(-21y) + \hat{\mathbf{y}}(10x) + 0\hat{\mathbf{z}}
\end{aligned}$$

Since the left side is equal to the right side, product rule (vi) is verified.